GAUS-AG: The Northcott Property in the $\overline{\mathbb{Q}}$-Isogeny Class of an Abelian Variety

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Let F be a number field with algebraic closure $\overline{F}$ and let A/F be an abelian variety. The Faltings height $h(A) \in \mathbb{R}$ is an invariant, which measures the arithmetic complexity of A. Faltings [Fal83] famously proved the abelian varieties in the isogeny class of A over F have bounded height and contain finitely many isomorphism classes. If one passes to the full $\overline{F}$-isogeny class of A, however, the fields of definition are no longer bounded, and Faltings' finiteness statement over a fixed field is not sufficient. The goal of the seminar is to study the recent theorem of Kisin and Mocz [KM26, Theorem 1.2].

Theorem. *Assume the Mumford–Tate conjecture holds for A. Let $\mathscr{I}(A)$ denote set of isomorphism classes of abelian varieties over $\overline{F}$ that are isogenous to $A_{\overline{F}}$. For any $c > 0$, the set*

$$\{B \in \mathscr{I}(A) \mid h(B) < c\}$$

is finite.

The proof studies the behavior of the Faltings height under isogeny using p-divisible groups and p-adic Hodge theory. The first part treats isogenies of p-power degree for a fixed prime p. The second part develops effective integral p-adic Hodge theoretic estimates which are uniform as p varies.

Time: The talks should take 60-90 minutes. We meet Thursdays at 13:30 on a weekly basis. Precise dates will be announced later.
Place: Frankfurt, Robert-Mayer-Str. 6-8, Raum 309
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Talks

Talk 1: The Northcott property and Faltings heights (Speaker: Benjamin)
Reference: [KM26, Introduction], [Fal83]
Recall the usual Northcott property for $\mathbb{P}^n_F$. Introduce the Faltings height and Faltings' finiteness theorems [Fal83]. Define and review the Faltings height [Fal83, p.354] and its basic properties. State and prove Faltings' isogeny formula [Fal83, Lemma 5]. State the main theorem of Kisin–Mocz and compare to Faltings. Explain the plan of the seminar.

Talk 2: p-divisible groups and Hodge-Tate decomposition (Speaker: Magdalena)
Reference: [KM26, §§2.1], [Tat67]
The goal of this talk is to provide some background on p-divisible groups necessary for [KM26, §§2.1] and the later talks. Review finite (flat) group schemes and the connected-étale exact sequence. Define p-divisible groups, their height, dimension and their Tate modules. When K is a complete, discrete valued field of mixed characteristic $(0, p)$ and perfect residue field and $C = \widehat{\overline{K}}$, define the C-points of a p-divisible groups and state the Hodge-Tate decomposition

for the Tate module of a p-divisible group over $\mathcal{O}_K$. State the Scholze-Weinstein classification [SW14, Theorem 5.2.1] of p-divisible groups over $\mathcal{O}_C$ and state/ prove [KM26, Lemma 2.1].

Talk 3: Local change of height (Speaker: Simon)
Reference: [KM26, §§2.2]
Let K be a complete, discrete valued field of mixed characteristic $(0,p)$ and perfect residue field. For a quasi-finite flat group scheme $\mathcal{G}/\mathcal{O}_K$ introduce the slope, height and degree $\mathcal{G}$ and define $\nu(\mathcal{H})$. Prove [KM26, Proposition 2.6 and 2.7].

Talk 4: Faltings' isogeny lemma (Speaker:)
Reference: [KM26, §§2.3.]
Let F be a number field. For a quasi-finite flat group scheme $\mathcal{H}/\mathcal{O}_F$, introduce the invariant $\nu(\mathcal{H})$. Express Faltings' isogeny formula in terms of $\nu(\mathcal{H})$ [KM26, Lemma 2.10] and prove [KM26, Lemma 2.11].

Talk 5: The Mumford–Tate conjecture for abelian varieties (Speaker:)
Reference: [Far16]
Define the Mumford-Tate group and formulate the Mumford-Tate Conjecture for abelian varieties. State Deligne's theorem [DMOS82, Theorem 2.11] and explain what it says about the Mumford-Tate Conjecture for abelian varieties [KM26, §§ 2.4, 2.4.1.]. Explain why the Mumford-Tate Conjecture (up to a finite extension) depends only on the isogeny class of $A_{\bar{F}}$. If time permits, state [CM20, Theorem A] and give an overview of the current state of the Mumford-Tate Conjecture.

Talk 6: Global change of height (Speaker:)
Reference: [KM26, §§2.4]
Let A/F be an abelian variety over a number field, such that the Mumford-Tate conjecture holds for A. Prove [KM26, Corollary 2.16], which describes the behavior of $h(A_{\overline{F}}/H[p^n])$ as $n \to \infty$, where $H \subset A_{\overline{F}}[p^\infty]$ is a p-divisible group

Talk 7: The Northcott property for p-isogenies (Speaker:)
Reference: [KM26, §§2.5]
The goal is to prove the main theorem for p-power isogeny classes [KM26, Theorem 2.20 and Corollary 2.21.]. Explain why the fixed-prime result is not sufficient for the full theorem: one must control kernels whose order is divisible by arbitrarily large primes.

Talk 8: φ-modules and slopes (Speaker:)
Reference: [KM26, §§3.1–3.2]
Introduce the φ-modules used in the paper, their degree and slope. Explain the compatibility between φ-modules and p-divisible groups [KM26, Proposition 3.7 and Corollary 3.8]

Talk 9: Fontaine–Laffaille theory and effective estimates (Speaker: Ruth)
Reference:[KM26, §§3.3–3.4]
Introduce Fontaine–Laffaille modules and prove the slope estimate [KM26, Corollary 3.19].

Talk 10: Modules with G-structure (Speaker:)
Reference:[KM26, §§3.5]
Let K be a complete, discrete valued field of mixed characteristic $(0,p)$ and perfect residue field. This talk is concerned with (crystalline) representations $\mathrm{Gal}_K \to G(\mathbb{Z}_p)$ where $G/\mathbb{Z}_p$ is a reductive group. The goal is to prove [KM26, Corollary 3.25.]

Talk 11: Proof of the Northcott theorem (Speaker:)
Reference: [KM26, §§3.6.]
Prove the main theorem [KM26, Theorem 3.3]. Since there is probably not enough time to provide enough background on Shimura varieties, it is okay to blackbox [KM26, Lemma 3.26].

References

[CM20] Anna Cadoret and Ben Moonen, *INTEGRAL AND ADELIC ASPECTS OF THE MUMFORD–TATE CONJECTURE*, J. Inst. Math. Jussieu **19** (2020), no. 3, 869–890.

[DMOS82] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih, *Hodge Cycles, Motives, and Shimura Varieties*, Lecture Notes in Mathematics, vol. 900, Springer, Berlin, Heidelberg, 1982.

[Fal83] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern.*, Invent. Math. **73** (1983), 349–366.

[Far16] Victoria Cantoral Farfán, *A survey around the Hodge, Tate and Mumford-Tate conjectures for abelian varieties*, February 2016.

[KM26] Mark Kisin and Lucia Mocz, *The Northcott Property in the $\bar{Q}$-Isogeny Class of an Abelian Variety*, Compos. Math. **162** (2026), no. 3, 703–742.

[SW14] Peter Scholze and Jared Weinstein, *Moduli of p-divisible groups*, Camb. J. Math. **1** (2014), no. 2, 145–237.

[Tat67] J. T. Tate, *P-Divisible Groups*, Proc. Conf. Local Fields (Berlin, Heidelberg) (T. A. Springer, ed.), Springer, 1967, pp. 158–183.