Block seminar on p-divisible groups, announcement

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Abstract

The main goal in Part I is to study results from Tate's seminal paper on p-divisible groups [Tat67], which initiated the field of p-adic Hodge theory by establishing the Hodge–Tate decomposition for abelian varieties. This part of the seminar will be fully accessible to master's students. If time and interest permits, we will also study the Serre–Tate theorem on deformations of abelian varieties in Part II. Then in Part III we will study applications of the earlier talks to the moduli of p-divisible groups in [SW13]. The seminar is primarily for master's students, although it is open to qualified bachelor's students and interested PhD students in case there are remaining timeslots. **Please write an email before 1. October** in case you are interested (*see section 3 below*). We encourage master's students to attend and emphasize that the advanced (Part III) talks are intended for PhD students or otherwise highly motivated master's students.

1 Overview of the seminar

Introduction and motivation One of the most powerful techniques in geometry is to attach linear algebraic invariants to spaces in a functorial way. For example, the singular and de Rham cohomologies of a smooth complex manifold X "understand" a great deal about the space. The de Rham comparison theorem can then be interpreted as providing an integral lattice $H^i_{\mathrm{sing}}(X,\mathbb{Z}) \subset H^i_{\mathrm{dR}}(X)$ that generates the de Rham cohomology as a complex vector space. If X is furthermore projective (and hence algebraic), then one can decompose the de Rham cohomology further into so-called Hodge classes

$$H^i_{\mathrm{dR}}(X) = \bigoplus_{p+q=n} H^{p,q}_{\mathrm{Hod}}(X),$$

where the Hodge cohomology classes of type (p,q) are determined by holomorphicity/anti-holomorphicity in local coordinates. The data $(H^\bullet_{\mathrm{sing}}(X,\mathbb{Z}), H^\bullet_{\mathrm{dR}}(X), H^{\bullet,\bullet}_{\mathrm{Hod}}(X))$ as above form the *Hodge structure* of X, which is the central object of complex Hodge theory.

The story of our seminar starts in the other direction and restricts to curves for simplicity: if X is an algebraic curve over $\operatorname{Spec}(\mathbb{Q})$ (that is, a smooth projective $\mathbb{Q}$-scheme of relative dimension 1), then to what extent is there a p-adic analogue of the Hodge structure for the compact Riemann surface $X(\mathbb{C})$? An algebraic analogue of the singular cohomology of X is given by the much more subtle p-adic étale cohomology $H^\bullet_{\text{ét}}(X, \mathbb{Z}_p) := \varprojlim_i H^\bullet_{\text{ét}}(X, \mathbb{Z}/p^i)$. Analogously to the complex curve setting, the étale cohomology of X is then completely determined by its Jacobian abelian variety $\operatorname{Jac}(X) = \operatorname{Pic}_0(X/\operatorname{Spec}\mathbb{Q})$, so that we shift our interest entirely to the case of abelian varieties A. The main goal of our seminar is to study the p-adic Hodge theory of abelian varieties with good reduction at p (that is, those whose p-completions admit a smooth integral model over $\mathbb{Z}_p$) in terms of *p-divisible groups* (sometimes referred to as *Barsotti–Tate* groups).

Part I: The Hodge–Tate decomposition Let p be a prime number, $\mathcal{O}$ a complete discrete valuation ring of mixed characteristic $(0, p)$ with residue field $k := \mathcal{O}/\mathfrak{m}$, and $K := \operatorname{Frac}\mathcal{O}$. Furthermore, write $C := (\overline{K})^\wedge_p$ for the p-adic completion of the algebraic closure of K. In particular, the Galois group $\operatorname{Gal}(\overline{K}/K)$ extends by continuity to an action on C and hence induces an action on the étale cohomology of any scheme over $\operatorname{Spec}(C)$.

Theorem 1 (Hodge–Tate decomposition for abelian varieties [Tat67, p.180 Remark])**.** *For any abelian scheme A over $\operatorname{Spec}(\mathcal{O})$ with base change $A_C = A \times_{\mathcal{O}} \operatorname{Spec} C$, there is a canonical decomposition of Galois modules*

$$H^1_{ét}(A_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \cong H^1_{ét}(A_C, \Omega^0_{A/C}) \oplus H^0_{ét}(A_C, \Omega^1_{A/C}(-1)).$$

The decomposition in Theorem 1 is simultaneously a p-adic analogue of the Hodge decomposition of complex de Rham cohomology, as well as the de Rham comparison theorem with singular cohomology. Our main goal is to understand Tate's proof (or modern reformulations) in terms of the p-divisible group $A[p^\infty] := \varinjlim_n A[p^n]$ over $\operatorname{Spf}(\mathbb{Z}_p)$ (where $A[p^n]$ denotes the p^n-torsion points) and the ramification theory of $K_\infty := \varinjlim_i K(\zeta_{p^i})$ over K (where ζ_{p^n} denotes a primitive p^n-th root of unity). Roughly, a p-divisible group over $\mathcal{O}$ (of height h) is a collection of finite $\mathcal{O}$-group schemes G_i of order ih together with injective maps $G_i \hookrightarrow G_{i+1}$ such that $G_i = \ker(G_{i+1} \xrightarrow{p^i\cdot} G_{i+1})$. We will learn about the structure of p-divisible groups G in terms of their connected subgroup G° and étale quotient $G^{\text{ét}}$, their Tate modules $T(G) := \varprojlim_i G_i(\overline{K})$, their Lie algebras as formal groups, and Cartier duality. Then the comparison in Theorem 1 follows from identifying the cohomology of A_C with the Tate module of A (equivalently of $A[p^\infty]$) and some Galois cohomology computations.

Part II: Deformation theory of abelian varieties Another goal is to study the infinitesimal local structure of the moduli of abelian varieties. More precisely, if $\overline{A}$ is an abelian variety over k and R is an Artinian local k-algebra, let $\operatorname{Def}_A(R)$ denote the category of deformations

$$\operatorname{Def}_{\overline{A}}(R) := \{(\widetilde{A}, \varepsilon) \mid \widetilde{A}/\operatorname{Spec} R \text{ is an abelian scheme and } \varepsilon : \widetilde{A} \otimes_R k \xrightarrow{\sim} \overline{A}\}.$$

One defines a similar category of deformations $\mathrm{Def}_{\overline{A}[p^\infty]}(R)$ in terms of p-divisible groups over R with reduction $\overline{A}[p^\infty]$.

Theorem 2 (Serre–Tate)**.** *The functor taking an abelian scheme $\widetilde{A}$ over R to its p-divisible group $\widetilde{A}[p^\infty]$ induces an equivalence*

$$\mathrm{Def}_{\overline{A}}(R) \xrightarrow{\sim} \mathrm{Def}_{\overline{A}[p^\infty]}(R),$$

functorially in Artinian local k-algebras R.

Along the way, one must prove that the deformation theory of $\overline{A}$ is *unobstructed* (i.e. existence of further deformations over R' for $\operatorname{Spec} R' \twoheadrightarrow \operatorname{Spec} R$, first as schemes and then as abelian varieties) using *rigidity* (i.e. the identity map on the special fiber must deform to the identity). Geometrically, that is to say that the moduli space of deformations of abelian varieties is formally smooth. Then the formal moduli problems are both compatibly (pro-)representable by a formal ball $W(k)[[t_1, \dots, t_{g^2}]]$ over the Witt vectors $W(k)$, where g is the dimension of $\overline{A}$ over k.

Part III: Moduli of p-divisible groups Finally, we wish to study analytic moduli of p-divisible groups (passing from formal schemes to adic spaces) in terms of their Hodge–Tate decompositions. For arbitrary p-divisible groups over $\mathcal{O}_C$, the decomposition in Theorem 1 takes the form of a (nonsplit) filtration

$$0 \to \mathrm{Lie}(G) \otimes C(1) \to T(G) \otimes C \to (\mathrm{Lie}(G^\vee))^\vee \otimes C \to 0.$$

Theorem 3 ([SW13, Theorem B])**.** *There is an equivalence of categories between the category of p-divisible groups over $\mathcal{O}_C$ and the category of finite rank free $\mathbb{Z}_p$-modules T together with a C-subvectorspace W of $T \otimes C(-1)$.*

Notably, this gives a *linear algebraic* description of p-divisible groups instead of a *Frobenius-linear* description via the Grothendieck–Messing theory of crystalline Dieudonné modules over $W(k)$. In this context, the Hodge–Tate filtration can be seen as a p-adic analogue of the conjugate filtration from complex Hodge theory, and Theorem 3 is an analogue of Riemann's classification of complex abelian varieties (in terms of their singular homology and the Hodge-filtration on the extension to $\mathbb{C}$).

To prove this for non-spherically complete C (e.g. $C = \mathbb{C}_p$), Scholze and Weinstein use crystalline period maps for (the adic generic fiber of) Rapoport–Zink's moduli space of deformations of p-divisible groups (which now allows the special fiber of the deformation to be an isogeny instead of an isomorphism) to relate the categories in Theorem 3 to certain modifications of vector bundles on the Fargues–Fontaine curve over C with infinite level structure [SW13, Theorem C]. This is the first example of a moduli space of so-called mixed-characteristic shtukas (after Drinfeld's equal-characteristic shtukas), which conceptually play the role of p-adic motives of good reduction over K.

The plan for this part of the seminar is to review Grothendieck–Messing's construction of Dieudonné crystals of p-divisible groups in terms of the Lie algebra of the universal vector extension, prove [SW13, Theorem A] on fully-faithfulness of the Dieudonné functor for p-divisible groups over certain semi-perfect rings, define the algebraic Fargues–Fontaine curve and describe its category of vector bundles in terms of isocrystals over $\overline{k}$, and deduce [SW13, Theorems B and C].

Outlook and further motivation There are many modern developments on the theory of p-divisible groups in recent years. One highlight is the PhD thesis of Lucas Gerth [Ger26], which defines analytic analogues of p-divisible groups that do not necessarily arise as the generic fiber of a formal p-divisible group and studies them in terms of their analytic/prismatic Dieudonné modules (e.g. [Ger26, Theorem 5.37]). Gerth furthermore proves that certain moduli of analytic p-divisible groups form moduli of p-adic shtukas [Ger26, Theorem 5.23], analogously to [SW13]. In another direction are the algebraicity results of Zachary Gardner and Keerthi Madapusi [GM26]. They use derived deformation theory and the Drinfeld/Bhatt–Lurie prismatic and syntomic stacks to confirm an algebraicity conjecture of Drinfeld on formal moduli of p-divisible groups (see [GM26, Theorems A and 9.3.2]). After nearly 60 years, the theory of p-divisible groups is still being developed in novel ways that are proving fundamental to modern p-adic geometry.

2 Time and place

The seminar will take place in February 2027 over three days (tentatively Fri. 19.2 to Sun. 21.2) in a Jugendherberge (youth hostel) that is reasonably accessible by regional trains from Frankfurt/Heidelberg. More details will appear in the coming months on the Uni-Frankfurt webpage of Jon Miles. There will be a **hybrid Zoom/in-person meeting on 12. October** to further discuss the dates and the distribution of talks.

3 Audience and prerequisites

The intended audience is primarily advanced master's students at GAUS universities[1] who are interested in algebraic geometry and number theory, and especially p-adic geometry. We will give remaining timeslots to PhD students at GAUS universities, depending on how many master's students are interested. If you are at all interested in attending the seminar (regardless of whether you are a bachelor's, master's, or PhD student), **please send an email** to Jon Miles[2] **before 1. October** with a short introduction of yourself. This should include 1. where you are studying and for which degree, 2. a brief list of relevant lectures/seminars you have attended, and 3. the lectures and seminars you plan to attend in WS26/27. It would also be appreciated if you could write a bit about the prerequisites from algebraic geometry and from number theory below. A large part of the

[1]The Rhein-Main universities TU Darmstadt, GU Frankfurt, JGU Mainz, as well as LU Hannover and Uni. Heidelberg (we will investigate options to provide credits outside the R-M system).

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seminar is determined by the background and number of the participants, so it is crucial that we receive your introductions sooner than later. Only the people who have sent an email will be invited to join the organizational meeting on 12. October.

The prerequisites are divided into three categories. Strict prerequisites (roughly Part I) should be known to all participants before the seminar begins at the end of the semester in February. Soft prerequisites (roughly Part II) should be known to several (but perhaps not all) participants in case we decide on a more advanced version of the seminar; otherwise, they will be the topics of some early talks. The Good to know column (roughly Part III) contains topics centered around the advanced talks at the end of the seminar (Dieudonné theory, moduli of p-divisible groups, and p-adic Hodge theory), and are good to at least be familiar with for following the talks. We will provide sources for the prerequisites in the seminar program.

Prerequisites from algebraic geometry:		
Strict prerequisites	Soft prerequisites	Good to know
· algebraic geometry I · (formally) étale maps and coverings · sheaves of relative Kähler differentials	· Abelian varieties/ Spec $\mathbb{Z}$ · Yoneda embedding and functor of points · formal schemes	· p-adic étale cohomology of curves · finite group schemes in positive characteristic · deformation theory · a *bit* of complex Hodge theory

Prerequisites from number theory and arithmetic geometry:		
Strict prerequisites	Soft prerequisites	Good to know
· p-adic numbers · infinite Galois theory · étale sheaves/field and cts. Galois modules	· Galois cohomology of p-adic fields · Witt vectors of perfect fields · torsion points of abelian varieties/elliptic curves	· adic and rigid spaces · Witt schemes of rings over perfect fields · crystalline cohomology

4 Further reading

We collect here some more informal writing on p-divisible groups and their motivation. These references should be used as a supplement for finding and reading the source material. The sources listed here and in the bibliography are far from a complete survey of the literature, but they are rather intended to help you decide if you would like to attend the seminar.

- The Bachelor's thesis of Tony Feng on the Hodge–Tate decomposition for abelian varieties [Fen13].

- A blog post by Alex Youcis on p-divisible groups, formal groups, and the Serre–Tate theorem [You15].
- A series of four blog posts by Akhil Mathew on deformations of abelian varieties and the Serre–Tate theorem [Mat12].
- Two Kleine AG block seminar programs on deformations of abelian varieties by Gregor Pohl and Nithi Rungtanapirom (auf deutsch) and by Thiago Solovera e Nery.

References

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[Ger26] Lucas Gerth. A dieudonné theory for analytic p-divisible groups and applications to shimura varieties. arXiv preprint, 2026.

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[SW13] Peter Scholze and Jared Weinstein. Moduli of p-divisible groups. *Cambridge Journal of Mathematics*, 1(2):145–237, 2013.

[Tat67] J. T. Tate. p-divisible groups. In T. A. Springer, editor, *Proceedings of a Conference on Local Fields*, pages 158–183, Berlin, Heidelberg, 1967. Springer Berlin Heidelberg.

[You15] Alex Youcis. p-divisible groups, formal groups, and the serre–tate theorem. `https://ayoucis.wordpress.com/2015/08/10/p-divisible-groups-formal-groups-and-the-serre-tate-theorem/`, 2015. blog post.