

Summer 2026
Dates: 30.04, 21.05, 28.05, 11.06.
10-12:30 Uhr, RM 6-8, Raum 309.

GAUS AG
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WKB Analysis: From Quadratic Differentials to Stokes matrices

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Abstract We will explore the interplay between the geometry of infinite-area quadratic differentials and the WKB analysis for the associated Schrödinger-type differential equations. This leads to an expression of the wild Riemann-Hilbert map, i.e. Stokes matrices, for $SL(2, \mathbb{C})$ -opers in terms of periods of the quadratic differential (so-called Voros periods). In the final session, we will discuss higher-rank opers and the conjectural extension of the WKB analysis using spectral networks.

1. Meromorphic differentials and their trajectories (60 min) 30.04. (NAME)

Introduce strata of meromorphic quadratic differentials, the canonical cover, hat-homology lattice and period coordinates [BS15, §2.2+§2.3] [Bai+19]. Discuss the trajectory structure at zeros and poles [BS15, §3.1-3.3] and the associated marked bordered surface [BS15, §6.1]. Show that for a saddle-free GMN differential there is a flat strip decomposition fixing a basis of the hat homology via the crossing/standard saddle connection [BS15, §3.4+3.5]. Explain how the horizontal strip decomposition of a saddle-free GMN differential induces an ideal triangulation of the marked bordered surface [BS15, Lemma 10.1].

References: [BS15] [Bai+19].

2. $\mathbb{C}P^1$ -structures and Schrödinger equations (60 min) 30.04. (NAME)

Define complex projective structures. Discuss developing map and monodromy. Define the affine structure over the space of quadratic differentials via Schrödinger equation and Schwarzian derivative [LMP09, §1][AB20, §2.1+2.2]. Explain that choosing a square-root of the canonical and a complex projective structure the Schrödinger equation can be globally defined a Riemann surface of positive genus. See [Dum+21, §1.2] for the statement. The local computation can be found in [IN14, §2.1]. Discuss the extension of the results to meromorphic projective structures [AB20, §3].

References: [LMP09] [AB20].

3. **WKB analysis 1: From Schrödinger equation to the Stokes graph** (60 min)
21.05. (NAME)

Introduce the Schrödinger-type equation associated to a quadratic differential on \mathbb{P}^1 [Iwa25, §1]. Explain how to construct the formal WKB solution using the Riccati equation Ansatz. Discuss [Iwa25, Example 1.2.] concerning Airy's equation – this will be our running example. Introduce the spectral curve for the quadratic differential, and its turning points (the branch points of the spectral curve). Define the Stokes graph [Iwa25, Definition 1.7.] and describe it in the Airy case.

References: [Iwa25].

4. **WKB analysis 2: Borel summation and summability of the WKB solution** (60 min)
21.05. (NAME)

The goal of this talk is to turn the formal WKB solution into analytic solutions away from the Stokes curves. Introduce the notion of Borel summation [Iwa25, §1.2.1.], and explain how it can be used to turn a formal solution into an analytic function. For further background (without parameters) see [PS03, §7.6.]. Illustrate Borel summation for the WKB solution of Airy's equation [Iwa25, Exercise 4]. Explain [Iwa25, Theorem 1.9.] about Borel summability of the WKB solution away from Stokes curves. Sketch the proof, in particular obstructions to Borel summability on Stokes curves, and coming from saddle connections.

References: [Iwa25] [PS03].

5. **Stokes matrices and Voros periods** (60 min) 28.05. (NAME)

In this talk we compare the exact solutions of the Schrödinger-type equation across different Stokes regions. Explain how to calculate the connection matrices (called Stokes matrices) for Stokes curves emanating from simple zeros [Iwa25, §1.3.1.], and from simple poles [Iwa25, §1.3.3.]. Define Voros periods [Iwa25, Definition 1.15.] and explain how the connection matrices can generally be computed in terms of Voros period integrals on the spectral curve.

References: [Iwa25].

6. **Gentle introduction to the irregular Riemann-Hilbert correspondence** (60 min)
28.05. (NAME)

The connection matrices from the previous talk are special cases of Stokes matrices, generalized monodromy data attached to connections with higher order poles. Recall the usual Riemann-Hilbert correspondence on a Riemann surface [PS03, Theorem §6.2., Example 6.6.], and sketch how to reconstruct a connection from a local system. Explain what breaks when you allow higher order poles using the exponential function and Airy's equation. Define singular directions for integrable meromorphic connections on a disk with pole order 2 and whose leading term has pairwise distinct eigenvalues [Boa01, §3]. Explain how to define the Stokes matrices [Boa01, Definition 16] using preferred fundamental solutions defined via resummation (use it as black box). For more background on resummation see also [PS03, §7.6.-7.8.]. State the irregular Riemann-Hilbert correspondence [Boa01, Theorem 5] in this case. Explain that the usual monodromy of the

connection can be recovered from the Stokes matrices and formal monodromy [Boa01, Lemma 18].

References: [PS03] [Boa01].

7. **SL(n, \mathbb{C})-Opers** (60 min) 30.04. (NAME)

Recall that the global Schrödinger equation on a Riemann surface is equivalent to a $SL(2, \mathbb{C})$ -oper [Dum+21, §1.2]. Define $SL(n, \mathbb{C})$ -opers [Dum+21, Def. 2.14][Wen16, §4.7]. The main goal is to describe the parametrization of connected components of opers with the $SL(n, \mathbb{C})$ -Hitchin base [Wen16, Theorem 4.17]. The construction of the associated $SL(n, \mathbb{C})$ -oper is given in [Dum+21, §2.10]. The full theorem is proven in [Wen16, §4.2+§4.3]. Outline the proof.

References: [Dum+21] [Wen16].

8. **SL(n, \mathbb{C})-spectral networks and WKB analysis for opers.** (60 min) 11.06. (NAME)

Introduce the spectral cover associated to a point in the meromorphic $SL(n, \mathbb{C})$ -Hitchin base. Introduce the root cover, solitons and the WKB spectral network [Nei, §2]. Explain the algorithmic construction of spectral networks [Nei, §2.5]. Note that in general spectral networks are infinite. Introduce [Nei, Conjecture 3.7] and highlight that the techniques of talk 3-5 give a proof for $n = 2$. If times allows, explain the path lifting rule for finite spectral networks as a homotopy invariant way to lift paths along a branched covering [Kin+26, §13.7][HM25, §4][Nei, §4]. Explain how the gluing of the exact solutions of the WKB analysis via Stokes matrices is an example of non-abelianisation [Nei, Prop. 4.4]. Compare to [Kin+26, §13] for a combinatorial view on spectral networks.

References: [Nei] [Kin+26], original source: [GMN13].

References

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